

# Hypersymmetry: a $\mathbb{Z}_3$ -graded generalization of Supersymmetry.

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## Abstract

We propose a generalization of non-commutative geometry and gauge theories based on ternary  $\mathbb{Z}_3$ -graded structures. In the new algebraic structures we define, we leave all products of two entities free, imposing relations on ternary products only. These relations reflect the action of the  $\mathbb{Z}_3$ -group, which may be either trivial, i.e.  $abc = bca = cab$ , generalizing the usual *commutativity*, or non-trivial, i.e.  $abc = jbc a$ , with  $j = e^{(2\pi i)/3}$ . The usual  $\mathbb{Z}_2$ -graded structures such as Grassmann, Lie and Clifford algebras are generalized to the  $\mathbb{Z}_3$ -graded case. Certain suggestions concerning the eventual use of these new structures in physics of elementary particles are exposed.

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# 1 Introduction

In a recent series of articles [3, 4, 5] we have advocated the  $\mathbb{Z}_3$ -grading as a natural generalization of well-known  $\mathbb{Z}_2$ -graded structures, such as graded Lie algebras, superspaces and  $\mathbb{Z}_2$ -graded generalizations of non-commutative geometry [1, 2].

Most of the cases in which the  $\mathbb{Z}_3$ -grading was studied has been based on the grading of ordinary algebras of matrices or operators.

In this article we wish to stress the fact that the natural structure on which the  $\mathbb{Z}_3$ -grading takes its full meaning is a *ternary algebra*, which means a linear vector space over complex numbers on which a *ternary composition law* is defined. Although ternary laws can be modelled in ordinary algebras with an associative binary law by defining corresponding ternary ideals and dividing the algebra by the equivalence relations induced by these ideals, one may also introduce ternary composition laws for the entities which can not be derived from a binary law.

Although we believe that these novel algebraic constructions might be pertinent for the description of quark fields and new models of elementary interactions in particle physics, we shall stress here mathematical rather than physical aspects, keeping hope that further developments and physical applications of ternary structures will follow soon.

During the last decade a spectacular development of non-commutative generalizations of differential geometry and Lie group theory has been achieved; the respective new chapters of mathematical physics are known under the names of *Non-Commutative Geometry* and *Quantum Groups and Quantum Spaces*. In both cases, the crucial question asked concerned the behavior of a “product”, or in more general terms, of a *binary composition law* under the permutation of two “factors”. We shall generalize this approach for the case in which no conditions are imposed on binary products (which may even not be defined in some cases),

but in contrast, specific behavior of ternary composition laws will be required.

In what follows, we shall briefly recall the action of the permutation group of three elements,  $S_3$ , in the complex plane, and apply its different representations to ternary composition laws. Next, we shall generalize the notions of Grassmann algebras, superspaces, supermatrices, Lie algebras and Clifford algebras replacing systematically binary symmetry conditions by their ternary counterparts.

Although many questions concerning the use of such ternary algebras in field theory remain unanswered, and a lot of constructions have still to be invented, we believe that this topic may attract attention of theoretical physicists.

## 2 The actions of $\mathbb{Z}_3$ and $S_3$ on ternary products.

The group of permutations of three objects contains six elements, three of which form the abelian subgroup  $\mathbb{Z}_3$ . These permutations are:

$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix} \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$

The first three elements represent *cyclic* permutations, including the identity. The entire group  $S_3$  can be generated by two elements, a cyclic permutation and one of the three involutions (odd permutations).

The important fact about the group  $S_3$  is that it is the last of permutation groups having a faithful representation in the complex plane; the next permutation group,  $S_4$ , containing 24 elements, has a representation in the complex plane that is not faithful, and starting from  $S_5$ , the permutation groups do not have representations in complex plane, besides the trivial and reduced representation assigning an involution to all odd elements and the identity to all even elements.

This fact is well known since Galois, and it is this very phenomenon that makes possible the existence of algebraic solutions (in complex numbers) of 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> order equations only, excluding the algebraic solutions of any equation

of order 5 and higher.

We shall represent the  $\mathbb{Z}_3$  group in the complex plane with the multiplications by the cubic roots of 1, i.e. 1,  $j$  and  $j^2$ , where

$$j = e^{2i\pi/3}, \quad j^2 = e^{-2i\pi/3}, \quad \text{and} \quad 1 + j + j^2 = 0$$

The odd permutations can be generated by complex conjugation; the remaining two odd permutations, corresponding to reflections in the roots  $j$  and  $j^2$  can be obtained via composition of complex conjugation with one of the cyclic elements (i.e. the rotations by  $2\pi/3$  and  $4\pi/3$ , represented as multiplications by  $j$  and  $j^2$  respectively).

In our generalizations of binary non-commutative structures we shall systematically replace the representations of the group  $\mathbb{Z}_2$  acting on binary relations by the representations of the group  $\mathbb{Z}_3$  acting on ternary relations. Whenever these relations can be represented as induced by an ordinary binary composition rule in some associative algebra, we shall suppose that the binary products are totally independent.

Of course, in our generalization we cannot distinguish the representations of  $\mathbb{Z}_2$  from that of the group  $S_2$ , because the groups are identical; here we have the choice between the cyclic group  $\mathbb{Z}_3$  or the whole permutation group  $S_3$  which has six elements. The resulting algebraic structures are very different too.

It is useful to remind that all binary relations can be interpreted in two alternative ways, depending on whether we write them on one side of the equation, or with non-trivial left- and right-hand sides; the ternary generalizations will impose stronger or weaker conditions when interpreted in these alternative manners.

Here are the examples of ternary generalizations of well-known binary structures that we shall study one by one in the next sections. We shall start with the classification of bi- (respectively, tri-linear) mappings from vector spaces into

complex numbers. The bilinear forms can be separated into different categories following their  $\mathbb{Z}_2$ -representation properties:

$$(X, Y) = (Y, X), \text{ a trivial representation of } \mathbb{Z}_2$$

which is called a symmetric 2-form. The same condition can be written as

$$(X, Y) + (-1)(Y, X) = 0$$

which in binary case turns out to be equivalent with the former one.

In the case of ternary generalization (3-linear forms satisfying given representation properties with respect to the group  $\mathbb{Z}_3$ ) similar conditions are no more equivalent:

$$(X, Y, Z) = (Y, Z, X) = (Z, X, Y), \text{ a trivial representation of } \mathbb{Z}_3,$$

the second interpretation will lead to the following condition:

$$(X, Y, Z) + j(Y, Z, X) + j^2(Z, X, Y) = 0$$

Here the first condition implies the second one, whereas from the second condition may follow either the first solution (i.e. all cyclic permutations being equal), or a second (and the only other possible) one, namely

$$(X, Y, Z) = j^2(Y, Z, X) = j(Z, X, Y)$$

which is stronger and obviously satisfies the above equation.

A similar scheme can be applied for the action of the whole  $S_3$  group if we decide how to represent the odd permutations. If only the action of  $\mathbb{Z}_3$  is represented, this means that the result of  $(Z, Y, X)$  is independent from  $(X, Y, Z)$ ; if not, the complex conjugation in complex plane provides us with a representation of  $S_3$  when combined with the rules for  $\mathbb{Z}_3$ :

$$(X, Y, Z) = \overline{(Z, Y, X)}$$

On the other hand, if we decide that  $(X, Y, Z) = (Z, Y, X)$ , this will also define an action of  $S_3$ , which in this case will not be faithful anymore.

The skew-symmetric bilinear forms are generalized by

$$(X, Y, Z) = j(Y, Z, X) = j^2(Z, X, Y)$$

or

$$(X, Y, Z) + (Y, Z, X) + (Z, X, Y) = 0$$

in the case of a  $\mathbb{Z}_3$ -generalization or

$$(X, Y, Z) + (Y, Z, X) + (Z, X, Y) + (Z, Y, X) + (Y, X, Z) + (X, Z, Y) = 0$$

in the case of the  $S_3$ -generalization.

### 3 $\mathbb{Z}_3$ -graded analogue of Grassmann algebra

Perhaps the simplest and the most straightforward  $\mathbb{Z}_3$ -graded generalization of a well-known binary algebraic structure is the Grassmann algebra.

Consider a finite-generated associative free algebra over complex numbers. Let us denote the generators of this algebra by  $\theta^A, \theta^B$ ,  $A, B = 1, 2, \dots, N$ . We suppose that the  $N^2$  products  $\theta^A\theta^B$  are linearly independent entities, whereas the products of three generators  $\theta^A\theta^B\theta^C$  are subjected to the following condition:

$$\theta^A\theta^B\theta^C = j\theta^B\theta^C\theta^A$$

The immediate corollary is that any product of four or more generators must vanish. Here is the proof:

$$(\theta^A\theta^B\theta^C)\theta^D = j\theta^B(\theta^C\theta^A\theta^D) = j^2(\theta^B\theta^A\theta^D)\theta^C = \theta^A(\theta^D\theta^B\theta^C) = j\theta^A\theta^B\theta^C\theta^D,$$

Now, as  $(1 - j) \neq 0$ , one must have  $\theta^A\theta^B\theta^C\theta^D = 0$ . The dimension of the  $\mathbb{Z}_3$ -graded Grassmann algebra is  $N(N + 1)(N + 2)/3 + 1$ . Any cube of a generator

is equal to zero; the odd permutation of factors in a product of three leads to an independent quantity.

Our algebra admits a natural  $\mathbb{Z}_3$ -grading: under multiplication, the grades add up modulo 3; the numbers are grade 0, the generators  $\theta^A$  are grade 1; the binary products are grade 2, and the ternary products grade 0 again. The dimensions of the subsets of grade 0, 1 and 2 are, respectively,  $N$  for grade 1,  $N^2$  for grade 2 and  $(N^3 - N)/3 + 1$  for grade 0.

The lack of symmetry between the grades 1 and 2 (corresponding to the generator  $j$  and its square  $j^2$  in the cyclic group  $\mathbb{Z}_3$ , which are interchangeable, suggests that one should introduce another set of  $N$  generators of grade 2, whose squares would be of grade 1, and which should obey the conjugate ternary relations as follows:

$$\bar{\theta}^{\bar{A}}\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}} = j^2\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}}\bar{\theta}^{\bar{A}}$$

With respect to the ordinary generators  $\theta^A$ , the conjugate ones should behave like the products of two  $\theta$ 's, i.e.

$$\theta^A(\theta^B\theta^C) = j(\theta^B\theta^C)\theta^A \rightarrow \theta^A\bar{\theta}^{\bar{B}} = j\bar{\theta}^{\bar{B}}\theta^A \quad (1)$$

and consequently,

$$\bar{\theta}^{\bar{B}}\theta^A = j^2\theta^A\bar{\theta}^{\bar{B}} \quad (2)$$

One may also note that there is an alternative choice for the commutation relation between the ordinary and conjugate generators, that makes the conjugate generators different from the binary products of ordinary generators:

$$\theta^A\bar{\theta}^{\bar{B}} = -j\bar{\theta}^{\bar{B}}\theta^A \text{ and } \bar{\theta}^{\bar{B}}\theta^A = -j^2\theta^A\bar{\theta}^{\bar{B}} \quad (3)$$

which are still compatible with the ternary relations introduced above.

This could be interpreted in the following way. We have assumed that the algebra's field is the field of complex numbers, but we can imagine that it is possible to multiply an element of the  $\mathbb{Z}_3$ -graded Grassmann algebra by an element

of a *binary* Grassmann algebra. We assume that the binary elements commute with the ternary ones, but anticommute as usual with each other. The  $\mathbb{Z}_3$ -graded Grassmann elements of a given grade still have no binary commutation relation. Then, our new algebra admits two gradings: the  $\mathbb{Z}_2$ -grading and the  $\mathbb{Z}_3$ -grading. The elements of  $\mathbb{Z}_2$ -grade 0 and  $\mathbb{Z}_3$ -grades 1 and 2 obey the rules (1) and (2) whereas the elements of  $\mathbb{Z}_2$ -grade 1 and  $\mathbb{Z}_3$ -grades 1 and 2 obey the rules (3). If we think that these objects can help in modelling of the quark fields, then a quark variable would be of  $\mathbb{Z}_2$ -grade 1 and  $\mathbb{Z}_3$ -grade 1, and an antiquark variable of  $\mathbb{Z}_2$ -grade 1 and  $\mathbb{Z}_3$ -grade 2. Then, the products of a quark and an antiquark would have both grades zero, making it a boson. In the same way, the products of three quark or three antiquark fields would be of  $\mathbb{Z}_3$ -grade 0 and of  $\mathbb{Z}_2$ -grade 1, that is, they would very much look like a fermionic field.

Now, the  $\bar{\theta}$ 's generate their own Grassmann subalgebra of the same dimension that the one generated by  $\theta$ 's; besides, we shall have all the mixed products containing both types of generators, but which can be always ordered e.g. with  $\theta^A$ 's in front and  $\bar{\theta}^{\bar{B}}$ 's in the rear, by virtue of commutation relations. The products of  $\theta^A$ 's alone or of  $\bar{\theta}^{\bar{A}}$ 's alone span two subalgebras of dimension  $N(N+1)(N+2)/3$  each; the mixed products span new sectors of the  $\mathbb{Z}_3$ -graded Grassmann algebra.

In the case of usual  $\mathbb{Z}_2$ -graded Grassmann algebras the anti-commutation between the generators of the algebra and the assumed associativity imply automatically the fact that *all* grade 0 elements *commute* with the rest of the algebra, while *any two* elements of grade 1 anti-commute.

In the case of the  $\mathbb{Z}_3$ -graded generalization such an extension of ternary and binary relations *does not follow automatically*, and must be explicitly imposed. If we decide to extend the relations (1), (2) and (3) to *all* elements of the algebra having a well-defined grade (i.e. the monomials in  $\theta$ 's and  $\bar{\theta}$ 's), then many additional expressions must vanish, e.g.:

$$\theta^A \underbrace{\theta^B \bar{\theta}^C}_{=0} = \underbrace{\theta^B \bar{\theta}^C}_{=0} \theta^A = \theta^B \underbrace{\bar{\theta}^C \theta^A}_{=0} = \bar{\theta}^C \theta^A \theta^B = 0$$

because on the one side,  $\theta^B \bar{\theta}^C$  and  $\bar{\theta}^C \theta^A$  are of grade 0 and commute with all other elements and on the other side, commuting  $\bar{\theta}^C$  with  $\theta^A \theta^B$  one gets twice the factor  $j$ , which leads to the overall factor  $j^2 \bar{\theta}^C \theta^A \theta^B$ . This produces a contradiction which can be solved only by supposing that  $\theta^A \theta^B \bar{\theta}^C = 0$ .

The resulting  $\mathbb{Z}_3$ -graded algebra contains only the following combinations of generators:

$$A_1 = \{\theta, \bar{\theta}\}; A_2 = \{\bar{\theta}, \theta\theta\}; A_0 = \{\mathbf{1}, \theta\bar{\theta}, \theta\theta\theta, \bar{\theta}\bar{\theta}\bar{\theta}\}$$

The dimension of the algebra is then

$$D(N) = 1 + 2N + 3N^2 + \frac{2(N^3 - N)}{3} = \frac{3 + 4N + 9N^2 + 2N^3}{3}$$

The four summands 1,  $2N$ ,  $3N^2$  and  $\frac{2(N^3 - N)}{3}$  correspond to the subspaces respectively spanned by the combinations  $\{\mathbb{C}\}$ ,  $\{\theta, \bar{\theta}\}$ ,  $\{\theta\theta, \theta\bar{\theta}, \bar{\theta}\bar{\theta}\}$  and  $\{\theta\theta\theta, \bar{\theta}\bar{\theta}\bar{\theta}\}$ .

Let us note that the set of grade 0 (which obviously forms a sub-algebra of the  $\mathbb{Z}_3$ -graded Grassmann algebra) contains the products which could symbolize the only observable combinations of *quark fields* in quantum chromodynamics based on the  $SU(3)$ -symmetry.

We can introduce the  $\mathbb{Z}_3$ -graded derivations of the  $\mathbb{Z}_3$ -graded Grassmann algebra by postulating the following set of rules:

$$\partial_A(\mathbf{1}) = 0; \quad \partial_A \theta^B = \delta_A^B; \quad \partial_A \bar{\theta}^{\bar{B}} = 0$$

and similarly

$$\partial_{\bar{A}}(\mathbf{1}) = 0; \quad \partial_{\bar{A}} \bar{\theta}^{\bar{C}} = \delta_{\bar{A}}^{\bar{C}}; \quad \partial_{\bar{A}} \theta^A = 0$$

When acting on various binary and ternary products, the derivation rules are the following:

$$\partial_A(\theta^B\theta^C) = \delta_A^B\theta^C + j\delta_A^C\theta^B; \quad \partial_A(\theta^B\theta^C\theta^D) = \delta_A^B\theta^C\theta^D + j\delta_A^C\theta^D\theta^B + j^2\delta_A^D\theta^B\theta^C$$

Similarly for the conjugate entities:

$$\partial_{\bar{A}}(\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}}) = \delta_{\bar{A}}^{\bar{B}}\bar{\theta}^{\bar{C}} + j^2\delta_{\bar{A}}^{\bar{C}}\bar{\theta}^{\bar{B}}; \quad \partial_{\bar{A}}(\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}}\bar{\theta}^{\bar{D}}) = \delta_{\bar{A}}^{\bar{B}}\bar{\theta}^{\bar{C}}\bar{\theta}^{\bar{D}} + j^2\delta_{\bar{A}}^{\bar{C}}\bar{\theta}^{\bar{D}}\bar{\theta}^{\bar{B}} + j\delta_{\bar{A}}^{\bar{D}}\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}}$$

Note the “twisted” Leibniz rule for the ternary products.

Finally, for mixed binary products like  $\theta^A\bar{\theta}^{\bar{B}}$  the derivation rules are as follows:

$$\partial_A(\theta^B\bar{\theta}^{\bar{C}}) = \delta_A^B\bar{\theta}^{\bar{C}}; \quad \partial_{\bar{A}}(\theta^B\bar{\theta}^{\bar{C}}) = j\delta_{\bar{A}}^{\bar{C}}\theta^B;$$

There is no need for rules of derivation of fourth-order homogeneous expressions, because these vanish identically.

As the immediate consequence of these rules we have the following important identities:

$$\partial_A\partial_B\partial_C = j\partial_B\partial_C\partial_A \text{ and } \partial_{\bar{A}}\partial_{\bar{B}}\partial_{\bar{C}} = j^2\partial_{\bar{B}}\partial_{\bar{C}}\partial_{\bar{A}}$$

while

$$\partial_A\partial_{\bar{C}} = j\partial_{\bar{C}}\partial_A \text{ and } \partial_{\bar{C}}\partial_A = j^2\partial_A\partial_{\bar{C}}$$

hence the important consequence

$$\partial_A\partial_B\partial_C + \partial_B\partial_C\partial_A + \partial_C\partial_A\partial_B = 0 \tag{4}$$

## 4 $\mathbb{Z}_3$ -graded algebra of hypersymmetry generators.

The  $\mathbb{Z}_3$ -graded generalization of the Grassmannian and the  $\mathbb{Z}_3$ -graded derivatives defined above can be used in order to produce a  $\mathbb{Z}_3$ -generalization of the supersymmetry generators acting on the usual  $\mathbb{Z}_2$ -graded Grassmann algebra generated

by anticommuting fermionic variables  $\theta^\alpha$  and  $\bar{\theta}^{\dot{\beta}}$ :

$$\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0, \quad \bar{\theta}^{\dot{\alpha}} \theta^{\dot{\beta}} + \theta^{\dot{\beta}} \bar{\theta}^{\dot{\alpha}} = 0, \quad \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} + \bar{\theta}^{\dot{\beta}} \bar{\theta}^{\dot{\alpha}} = 0$$

With the “anti-Leibniz” rule of derivation

$$\partial_\alpha(\theta^\beta \theta^\gamma) = \delta_\alpha^\beta \theta^\gamma - \delta_\alpha^\gamma \theta^\beta$$

and similarly for any two dotted indices or mixed indices, one verifies easily that all such derivations do anticommute:

$$\partial_\alpha \partial_\beta + \partial_\beta \partial_\alpha = 0, \quad \partial_{\dot{\alpha}} \partial_{\dot{\beta}} + \partial_{\dot{\beta}} \partial_{\dot{\alpha}} = 0, \quad \partial_\alpha \partial_{\dot{\beta}} + \partial_{\dot{\beta}} \partial_\alpha = 0$$

These rules enable us to construct the generators of the supersymmetric (or  $\mathbb{Z}_2$ -graded) “odd” translations:

$$\mathcal{D}_\alpha = \partial_\alpha + \sigma_{\alpha\dot{\beta}}^k \bar{\theta}^{\dot{\beta}} \partial_k, \quad \mathcal{D}_{\dot{\beta}} = \partial_{\dot{\beta}} + \sigma_{\alpha\dot{\beta}}^m \theta^\alpha \partial_m$$

where both dotted and un-dotted indices  $\alpha, \dot{\beta}$  take the values 1 and 2, while the space-time indices k,l,m run from 0 to 3. The anti-commutators of these differential operators yield the ordinary (“even”) space-time translations:

$$\mathcal{D}_\alpha \mathcal{D}_{\dot{\beta}} + \mathcal{D}_{\dot{\beta}} \mathcal{D}_\alpha = 2 \sigma_{\alpha\dot{\beta}}^k \partial_k$$

while

$$\mathcal{D}_\alpha \mathcal{D}_\beta + \mathcal{D}_\beta \mathcal{D}_\alpha = 0, \quad \mathcal{D}_{\dot{\alpha}} \mathcal{D}_{\dot{\beta}} + \mathcal{D}_{\dot{\beta}} \mathcal{D}_{\dot{\alpha}} = 0$$

The  $\mathbb{Z}_3$ -graded generalization would amount to find a “cubic root” of a linear differential operator, making use of equation 4. We must have six kinds of generalized Grassmann variables  $\theta^A, \theta^{\dot{A}}, \theta^{\dot{\dot{A}}}$  on the one hand and  $\bar{\theta}^{\bar{A}}, \bar{\theta}^{\dot{\bar{A}}}, \bar{\theta}^{\dot{\dot{\bar{A}}}}$  on the other hand, which is formally analogous to the  $\mathbb{Z}_2$ -graded case. All kinds of  $\theta$ ’s and  $\bar{\theta}$ ’s act like those that were introduced in section 3. Instead of the Pauli

matrices we should introduce the entities endowed with three indices (“cubic matrices”) with which the generators of the  $\mathbb{Z}_3$ -graded translations of grade 1 and 2 may be constructed as follows:

$$\begin{aligned} D_A &= \partial_A + \rho_{ABC}^m \theta^{\hat{A}} \theta^{\hat{C}} \nabla_m + \omega_{A\bar{A}}^m \bar{\theta}^{\bar{A}} \nabla_m, \quad D_{\bar{A}} = \partial_{\bar{A}} + \bar{\rho}_{\bar{A}\bar{B}\bar{C}}^m \bar{\theta}^{\hat{\bar{B}}} \bar{\theta}^{\hat{\bar{C}}} \nabla_m + \bar{\omega}_{A\bar{A}}^m \theta^A \nabla_m \\ D_{\hat{B}} &= \partial_{\hat{B}} + \rho_{ABC}^m \theta^A \theta^{\hat{C}} \nabla_m + \omega_{\hat{B}\hat{B}}^m \bar{\theta}^{\hat{B}} \nabla_m, \quad D_{\hat{\bar{B}}} = \partial_{\hat{\bar{B}}} + \bar{\rho}_{\hat{A}\hat{B}\hat{C}}^m \bar{\theta}^{\hat{\bar{A}}} \bar{\theta}^{\hat{\bar{C}}} \nabla_m + \bar{\omega}_{\hat{B}\hat{B}}^m \theta^{\hat{B}} \nabla_m \\ D_{\hat{C}} &= \partial_{\hat{C}} + \rho_{ABC}^m \theta^A \theta^{\hat{B}} \nabla_m + \omega_{\hat{C}\hat{C}}^m \bar{\theta}^{\hat{C}} \nabla_m, \quad D_{\hat{\bar{C}}} = \partial_{\hat{\bar{C}}} + \bar{\rho}_{\hat{A}\hat{B}\hat{C}}^m \bar{\theta}^{\hat{\bar{A}}} \bar{\theta}^{\hat{\bar{B}}} \nabla_m + \bar{\omega}_{\hat{C}\hat{C}}^m \theta^{\hat{C}} \nabla_m \end{aligned}$$

The nature of the indices needs not to be specified; the only important thing to be assumed at this stage is that the differential operators  $\nabla_m$  do commute with the  $\mathbb{Z}_3$ -graded differentiations  $\partial_A$ . It is also interesting to consider the operators one gets when the  $\nabla_m$  are replaced with *supersymmetric* derivations (that anti-commute with the  $\mathbb{Z}_3$ -graded differentiations). But in the simpler case described here, the following operators acting on the  $\mathbb{Z}_3$ -graded generalized Grassmannian:

$$\begin{aligned} D_{ABC}^{III} &= D_A D_B D_C + D_B D_C D_A + D_C D_A D_B + D_C D_B D_A + D_B D_A D_C + D_A D_C D_B \\ \bar{D}_{\bar{A}\bar{B}\bar{C}}^{III} &= D_{\bar{A}} D_{\bar{B}} D_{\bar{C}} + D_{\bar{B}} D_{\bar{C}} D_{\bar{A}} + D_{\bar{C}} D_{\bar{A}} D_{\bar{B}} + D_{\bar{C}} D_{\bar{B}} D_{\bar{A}} + D_{\bar{B}} D_{\bar{A}} D_{\bar{C}} + D_{\bar{A}} D_{\bar{C}} D_{\bar{B}} \\ D_{AA}^{II} &= D_A D_{\bar{A}} - j^2 D_{\bar{A}} D_A \end{aligned}$$

represent *homogeneous* operators on the  $\mathbb{Z}_3$ -graded Grassmann algebra, i.e. they map polynomials in  $\theta$ 's of a given grade into polynomials of the same grade; the result can be represented by a complex-valued matrix containing various combinations of the differentiations  $\nabla_m$ ; their eventual symmetry properties will depend on the assumed symmetry properties of the matrices  $\rho_{ABC}$  and  $\omega_{A\bar{A}}$ .

Let us consider in more detail the case of dimension 3 (the simplest possible realization of the  $\mathbb{Z}_3$ -graded Grassmannian and the derivations on it is of course the case with *one* generator and its conjugate; this has been considered in a paper by Won-Sang Chung[6]).

The dimension of the  $\mathbb{Z}_3$ -graded Grassmann algebra with three grade-1 generators  $\theta$ ,  $\hat{\theta}$  and  $\check{\theta}$  and three “conjugate” grade-2 generators  $\bar{\theta}$ ,  $\hat{\bar{\theta}}$  and  $\check{\bar{\theta}}$  is 51; any linear operator, including the derivations  $\partial_A$  and the multiplication by any combination of the generators, as well as the operators  $D_A$  and  $D_{\bar{A}}$  introduced above, can be represented by means of  $51 \times 51$  complex-valued matrices. Unfortunately, the operators  $D^{II}$  and  $D^{III}$  are neither diagonal nor diagonalizable. But if we apply them to a scalar function  $f$ , we get:

$$D_{1\bar{1}}^{II}f = (\omega_{1\bar{1}}^m + \bar{\omega}_{\bar{1}1}^m)\nabla_m f$$

and

$$D_{1\bar{1}\bar{1}}^{III}f = -3j^2\rho_{1\bar{1}\bar{1}}^m\nabla_m f \text{ as well as } \bar{D}_{\bar{1}\bar{1}\bar{1}}^{III}f = -3j\bar{\rho}_{\bar{1}\bar{1}\bar{1}}^m\nabla_m f$$

The  $\omega$  matrices are the only ones that remain in the  $D^{II}$  whereas the  $\rho$  cubic matrices emerge from the ternary combinations  $D^{III}$ . On the space of scalar functions, our operators act simultaneously as *square* and *cubic* roots of ordinary translations. Using extensions of these objects where the  $\nabla_m$  are replaced with the supersymmetry generators, we have contructed a simple  $\mathbb{Z}_3$ -graded non commutative geometry model featuring three Higgs fields. The lagrangian contains the potential term of degree 6:

$$V = 3|\Phi_1 + \Phi_2 + \Phi_3 + \Phi_1\Phi_2 + \Phi_2\Phi_3 + \Phi_3\Phi_1 + \Phi_1\Phi_2\Phi_3|^2$$

and implies multiple spontaneous symmetry breaking. This model will be the subject of another article.

## 5 $\mathbb{Z}_3$ -graded matrices.

If we want to have an integration theory on “hypermanifolds”, we will need the equivalents of matrices and determinant that should naturally appear in the formula of change of variables. If we align the basis of our  $\mathbb{Z}_3$ -graded Grassmann

algebra, with all the elements of grade 0 first, then all the elements of grade 1 and finally the elements of grade 2 in a one-column vector, any linear transformation that would leave these entries in the same order can be symbolized like the  $D^{II}$  and  $D^{III}$  operators by a matrix whose entries have a definite  $\mathbb{Z}_3$ -grade placed as follows:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

then the position of the three grades does not change in the resulting column; we shall call any matrix displaying this block grading structure a *grade 0* matrix. We can introduce two other kinds of matrices that raise all the grades by 1 resp. by 2, like the  $D_A$  resp.  $D_{\bar{A}}$  operators, as follows and calling them respectively *grade 1* and *grade 2* matrices:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

(the numbers 0, 1, 2 symbolizing the grades of the respective entries in the matrices). These matrices have been studied in detail in [9].

It is easy to check that the grades of these matrices add up modulo 3 under the associative matrix multiplication law.

One can define the analogues of the *supertrace* and *superdeterminant* of a matrix

$$M = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

as follows:

$$\mathrm{tr}_{\mathbb{Z}_3}(M) = \mathrm{tr} A + j^{2(1-\mathrm{gr}(M))} \mathrm{tr} E + j^{(1-\mathrm{gr}(M))} \mathrm{tr} I$$

and, for grade 0 matrices:

$$\det_{\mathbb{Z}_3}(M) = \det(A - CI^{-1}G - (B - CI^{-1}H)(E - FI^{-1}H)^{-1}(D - FI^{-1}G)) \times \\ \times (\det(E - FI^{-1}H))^{j^2} (\det I)^j$$

There are five other equivalent formulations of the  $\mathbb{Z}_3$ -determinant reflecting the six-fold  $S_3$  symmetry, just as there are two alternative formulations of the superdeterminant that reflect the  $S_2$  symmetry[7].

The  $\mathbb{Z}_3$  trace and determinant satisfy all the usual properties one would expect, especially the most important one:

$$\det_{\mathbb{Z}_3}(\exp(M)) = \exp(\text{tr}_{\mathbb{Z}_3}(M))$$

We expect this determinant to play the same role in integration theory on “hypermanifolds” as the superdeterminant in integration on supermanifolds.

## 6 The $\mathbb{Z}_3$ -graded exterior differential.

Consider the algebra  $M_3(\mathbb{C})$  of  $3 \times 3$  complex matrices, with the  $\mathbb{Z}_3$ -grading introduced above. Let  $B, C$  denote two matrices whose grades are  $\text{grad}(A) = a$  and  $\text{grad}(B) = b$ , respectively. We define the  $\mathbb{Z}_3$ -graded commutator  $[B, C]$  as follows:

$$[B, C]_{\mathbb{Z}_3} := BC - j^{bc}CB,$$

(Note that this  $\mathbb{Z}_3$ -graded commutator does not satisfy the Jacobi identity). Let  $\eta$  be a matrix of grade 1; we can choose for the sake of simplicity

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

With the help of the matrix  $\eta$  we can define a formal “differential” on the  $\mathbb{Z}_3$ -graded algebra of  $3 \times 3$  matrices as follows:

$$dB := [\eta, B]_{\mathbb{Z}_3} = \eta B - j^b B \eta$$

It is easy to show that  $d(BC) = (dB)C + j^b B(dC)$  and that  $d^3 = 0$ . It is also possible to define an external differential with a similar property, *i.e.*  $d^2 \neq 0$ ,  $d^3 = 0$ , on a manifold.

Let  $M_n$  be a differentiable manifold of dimension  $N$ , with local coordinates  $x^k$ . The variables  $x^k$  commute and their  $\mathbb{Z}_3$ -grade is 0. The linear operator  $d$  applied to  $x^k$  produces a 1-form whose  $\mathbb{Z}_3$ -grade is 1 by definition; when applied two times by iteration, it produces a new form of grade 2, denoted by  $d^2x^k$ . We shall postulate  $d^3 = 0$ .

Let  $F(M)$  denote the algebra of functions  $C^\infty(M)$ , over which the  $\mathbb{Z}_3$ -graded algebra generated by the forms  $dx^i$  and  $d^2x^k$  behaves as a *left* module. In other words, we shall be able to multiply the forms  $dx^i$ ,  $d^2x^k$ ,  $dx^i dx^k$ , etc. by the functions *only on the left*; right multiplication will just not be considered here. That is why we will write by definition, e.g.

$$d(x^i x^k) := x^i dx^k + x^k dx^i$$

We shall also assume the following Leibniz rule for the operator  $d$  with respect to the multiplication of the  $\mathbb{Z}_3$ -graded forms: when  $d$  crosses a form of grade  $p$ , the factor  $j^p$  appears as follows:

$$d(\omega \phi) = (d\omega)\phi + j^p \omega(d\phi)$$

Let us note that in contrast with the  $\mathbb{Z}_2$ -graded case, the forms are treated as one whole, even when multiplied from the left by an arbitrary function; that means that we can not identify e.g.  $(\omega_i dx^i)(\phi_k dx^k)$  with  $(\omega_i \phi_k)dx^i dx^k$

This is equivalent with saying that the products of functions by the forms are to be understood in the sense of tensor products, which is associative, but non-commutative.

Nevertheless, such an identification can be done for the forms of maximal degree (i.e. 3), which contain the products of the type  $dx^i dx^k dx^m$  or  $dx^i d^2x^m$ , whose exterior differentials vanish disrespectsfully of the order of the multiplication.

With the so established  $\mathbb{Z}_3$ -graded Leibniz rule, the postulate  $d^3 = 0$  suggests in an almost unique way the ternary and binary commutation rules for the differ-

entials  $dx^i$  et  $d^2x^k$ . To begin with, consider the differentials of a function of the coordinates  $x^k$ , with the “first differential”  $df$  that coincides with the usual one:

$$\begin{aligned} df & : = (\partial_i f) dx^i \\ d^2 f & : = (\partial_k \partial_i f) dx^k dx^i + (\partial_i f) d^2 x^i \\ d^3 f & = (\partial_m \partial_k \partial_i f) dx^m dx^k dx^i + (\partial_k \partial_i f) d^2 x^k dx^i + j(\partial_k \partial_i f) dx^i d^2 x^k + (\partial_k \partial_i f) dx^k d^2 x^i \end{aligned}$$

(we remind that the last part of the differential,  $(\partial_i f) d^3 x^i$ , vanishes by virtue of the postulate  $d^3 x^i = 0$ ).

Supposing that the partial derivatives commute, exchanging the summation indices  $i$  et  $k$  in the last expression and replacing  $1 + j$  by  $-j^2$ , we arrive at the following two independant conditions that lead to the vanishing of  $d^3 f$  :

$$dx^m dx^k dx^i + dx^k dx^i dx^m + dx^i dx^m dx^k = 0 \quad \text{and} \quad d^2 x^k dx^i - j^2 dx^i d^2 x^k = 0 .$$

which lead in turn to the following choice of relations (which of course is not unique):

$$dx^i dx^k dx^m = j dx^k dx^m dx^i \quad \text{and} \quad dx^i d^2 x^k = j d^2 x^k dx^i .$$

By extending these rules to *all* the expressions with a well-defined grade, and applying the associativity of the  $\mathbb{Z}_3$ -exterior product, it is easy to verify that all the expressions of the type  $dx^i dx^k dx^m dx^n$  and  $dx^i dx^k d^2 x^m$  must vanish, and along with these, also the monomials of higher order that would contain them as factors. Still, this is not sufficient in order to satisfy the rule  $d^3 = 0$  on all the forms spanned by the generators  $dx^k$  and  $d^2 x^k$ . It can be proved without much effort that the expressions containing  $d^2 x^i d^2 x^k$  must vanish, too. For example, if we take the particular 1-form  $x^i dx^k$  and apply to it the operator  $d$ , we get

$$d(x^i dx^k) = dx^i dx^k + x^i d^2 x^k ;$$

$$d^2(x^i dx^k) = d^2x^i dx^k + (1+j)dx^i d^2x^k = d^2x^i dx^k - d^2x^k dx^i;$$

which leads to  $d^3(x^i dx^k) = d^2x^i d^2x^k - d^2x^k d^2x^i$ . There is another possibility which is to say that the  $d^2x^k$  should commute with one another<sup>1</sup>. This makes the differential algebra infinite, but prevents us from considering the  $d^2x^k$  as grade 2 entities. But if we want to keep both the associativity of the “exterior product” and the ternary rule for the entities of grade 2, i.e.  $d^2x^i d^2x^k d^2x^m = j^2 d^2x^k d^2x^m d^2x^i$ , then the only solution is to impose  $d^2x^i d^2x^k = 0$  and to set forward the additional rule declaring that any expression containing *fourth or higher* power of the operator  $d$  must vanish identically.

With the above set of rules we can check that  $d^3 = 0$  on all the forms, whatever their grade or degree. Let us show how such calculus works on the example of a 1-form  $\omega = \omega_k dx^k$ :

$$d(\omega_k dx^k) = (\partial_i \omega_k) dx^i dx^k + \omega_k d^2x^k;$$

$$d^2(\omega_k dx^k) = (\partial_m \partial_i \omega_k) dx^m dx^i dx^k + (\partial_i \omega_k) d^2x^i dx^k + j dx^i d^2x^k + \partial_i \omega_k dx^i d^2x^k;$$

after exchanging the summation indices  $i$  and  $k$  in two last terms and using the fact that  $j+1 = -j^2$  and the commutation relations between  $dx^k$  and  $d^2x^i$ , we can write

$$d^2(\omega_k dx^k) = (\partial_m \partial_i \omega_k) dx^m dx^i dx^k + (\partial_i \omega_k - \partial_k \omega_i) d^2x^i dx^k.$$

where it is interesting to note how the usual *anti-symmetric* exterior differential appears as a part of the whole expression.

It is also easy to check that  $Im(d) \subseteq Ker(d^2)$  and  $Im(d^2) \subseteq Ker(d)$ .

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<sup>1</sup>this has been suggested by C. Juszczak

## 7 $\mathbb{Z}_3$ -graded gauge theory.

Let  $A$  be an associative algebra with unit element, and let  $H$  be a free left module over this algebra. Let  $A$  be a  $A$ -valued 1-form defined on a differential manifold  $M$ , and let  $\Phi$  be a function on the manifold  $M$  with values in the module  $H$ .

We shall introduce the *covariant differential* as usual:

$$D\Phi := d\Phi + A\Phi;$$

If the module is a free one, any of its elements  $\Phi$  can be represented by an appropriate element of the algebra acting on a fixed element of  $H$ , so that one can always write  $\Phi = B\Phi_o$ ; then the action of the group of automorphisms of  $H$  can be translated as the action of the same group on the algebra  $A$ .

Let  $U$  be a function defined on  $M$  with its values in the group of the automorphisms of  $H$ . The definition of a covariant differential is equivalent with the requirement  $DU^{-1}B = U^{-1}DB$ ; as in the usual case, this leads to the following well-known transformation for the connection 1-form  $A$  :

$$A \Rightarrow U^{-1}AU + U^{-1}dU;$$

But here, unlike in the usual theory, the second covariant differential  $D^2\Phi$  is not an automorphism: as a matter of fact, we have:

$$D^2\Phi = d(d\Phi + A\Phi) + A(d\Phi + A\Phi) = d^2\Phi + dA\Phi + jAd\Phi + Ad\Phi + A^2\Phi;$$

the expression containing  $d^2\Phi$  et  $d\Phi$ ; whereas  $D^3\Phi$  is an automorphism indeed, because it contains only  $\Phi$  multiplied on the left by an algebra-valued 3-form:

$$D^3\Phi = (d^2A + d(A^2) + AdA + A^3)\Phi = (D^2A)\Phi := \Omega\Phi;$$

Obviously, because  $D(U^{-1}\Phi) = U^{-1}(D\Phi)$ , one also has:

$D^3(U^{-1}\Phi) = U^{-1}(D^3\Phi) = U^{-1}\Omega\Phi = U^{-1}\Omega UU^{-1}\Phi$ , which proves that the 3-form  $\Omega$  transforms as usual,  $\Omega \mapsto U^{-1}\Omega U$  when the connection 1-form transforms according to the law:  $A \mapsto U^{-1}AU + U^{-1}dU$ .

It can be also proved by a direct calculus that the curvature 3-form  $\Omega$  does vanish identically for  $A = U^{-1}dU$  (see [8]).

It is interesting to compute the expression of the curvature 3-form in local coordinates:

$$\Omega = d^2A + d(A^2) + AdA + A^3 = \Omega_{ikm}dx^i dx^k dx^m + F_{ik}d^2x^i dx^k$$

where

$$\Omega_{ikm} := \partial_i \partial_k A_m + A_i \partial_k A_m - \partial_k A_m A_i + A_i A_k A_m \quad \text{and} \quad F_{ik} := \partial_i A_k - \partial_k A_i + A_i A_k - A_k A_i$$

In  $F_{ik}$  one can easily recognize the 2-form of curvature of the *usual* gauge theories.

We know that the expression  $F_{ik}$  is covariant with respect to the gauge transformations; on the other hand, the 3-form  $\Omega$  is also covariant; therefore, the local expression  $\Omega_{ijk}$  must be covariant, too. As a matter of fact, it can be expressed as a combination of covariant derivatives of the 2-form  $F_{ik}$ :

$$\Omega_{ikm} = \frac{1}{3}[j D_i F_{mk} + j^2 D_k F_{mi}],$$

or, equivalently,

$$\Omega_{ikm} = -\frac{1}{6}[D_i F_{mk} + D_k F_{mi}] + \frac{i\sqrt{3}}{6}[D_i F_{mk} - D_k F_{mi}]$$

The natural symmetry between  $j$  et  $j^2$ , which leads to the possibility of choosing one of these two complex numbers as the generator of the group  $\mathbb{Z}_3$ , and simultaneous interchanging the roles between the grades 1 and 2, suggests that we could extend the notion of complex conjugation  $j \Rightarrow (j)^* := j^2$ , with  $((j)^*)^* = j$ , to the algebra of  $\mathbb{Z}_3$ -graded exterior forms and the operator  $d$  itself.

It does not seem reasonable to use the “second differentials”  $d^2x^i$  as the objects conjugate to the “first differentials”  $dx^i$ , because the rules of  $\mathbb{Z}_3$ -graded exterior differentiation we have imposed break the symmetry between these two kinds of differentials: remember that the products  $dx^i dx^k$ , and  $dx^i dx^k dx^m$  are admitted, while we require that  $d^2x^i d^2x^k$  and  $d^2x^i d^2x^k d^2x^m$  must vanish.

This suggests the introduction of a “*conjugate*” differential  $\delta$  of grade 2, the image of the differential  $d$  under the conjugation  $*$ , satisfying the following conjugate relations:

$$\delta x^i \delta x^k \delta x^m = j^2 \delta x^k \delta x^m \delta x^i, \quad \delta x^i \delta^2 x^k = j^2 \delta^2 x^k \delta x^i .$$

All the relations existing between the operator  $d$  and the exterior forms generated by  $dx^i$  and  $d^2x^k$  are faithfully reproduced under the conjugation  $*$  if we consider the  $\mathbb{Z}_3$ -graded algebra generated by the entities  $\delta x^i$  and  $\delta^2 x^k$  as a *right module* over the algebra of functions  $F(M)$ , with the operator  $\delta$  acting *on the right* on this module.

The rules  $d^3 = 0$  and  $\delta^3 = 0$  suggest their natural extension:

$$d\delta = \delta d = 0$$

We would like to be able to form quadratic expressions that could define a scalar product; to do this, we should postulate that the algebra generated by the elements  $dx^i$ ,  $d^2x^k$  and its conjugate algebra generated by the elements  $\delta x^i$ ,  $\delta x^k$  commute with each other.

Then, we can define scalar products for the forms of maximal degree 3:  $\langle \omega | \phi \rangle := * \omega \phi$ , and integrating this result with respect to the usual volume element defined on the manifold  $M$ , which gives explicitly:

$$\int \bar{\omega}_{ikm} \phi_{prs} \langle \delta x^i \delta x^k \delta x^m | dx^p dx^r dx^s \rangle \quad \text{and} \quad \int \bar{\psi}_{ik} \chi_{mn} \langle \delta x^i \delta^2 x^k | d^2 x^m dx^n \rangle$$

What remains now is to determine the scalar products of the basis of forms; in order to assure the hermiticity of the product, one can always choose an “orthonormal” basis in which we should have:

$$\langle \delta x^i \delta x^k \delta x^m | dx^p dx^r dx^s \rangle = \delta_s^i \delta_r^k \delta_p^m \quad \text{and} \quad \langle \delta x^i \delta^2 x^k | d^2 x^m dx^n \rangle = \mu \delta_n^i \delta_m^k.$$

Here the first scalar product is normed to 1, and  $\mu$  is the ratio between the two types of “elementary volume”.

We consider the two types of forms of degree 3,  $dx^i dx^k dx^m$  and  $d^2 x^p dx^r$ , as being mutually orthogonal.

The above scalar product and generalized integral of  $\mathbb{Z}_3$ -graded forms enable us to introduce the Lagrangian densities involving quadratic expressions in both  $F_{ik}$  and  $\Omega_{ikl}$ .

## 8 A ternary generalization of Lie algebras.

Once the “ternary logic” is adopted, it can be quite easily applied to other well-known algebraic concepts, e.g. Lie algebras. There are other ternary generalizations of Lie algebra (see for example [10, 11, 12, 13]), from which this one is quite different, as far as we know.

The well-known Ado’s theorem states that any finite-dimensional Lie algebra can be realized with the elements of an associative algebra (the so-called enveloping algebra, whose dimension is usually much bigger than the dimension of the Lie algebra we started with) so that the skew-symmetric (and non-associative) multiplication law of the Lie algebra is replaced by the commutator  $[A, B] = AB - BA$ . The Jacobi identity, which is an independent postulate in the definition of an abstract Lie algebra, follows then automatically due to the associativity in the enveloping algebra.

The skew-symmetric composition law defined by a commutator in an associative algebra can be readily and naturally generalized as follows:

$$\{X, Y, Z\} = XYZ + jYZX + j^2ZXY + ZYX + j^2YXZ + jXZY$$

The ternary “3-commutator” thus defined satisfies the following properties corresponding to the antisymmetry property in the usual case:

$$\{X, Y, Z\} = j\{Y, Z, X\} = j^2\{Z, X, Y\}$$

from which follow the *two* identities:

$$\{X, Y, Z\} + \{Y, Z, X\} + \{Z, X, Y\} = 0 \quad \text{and} \quad \{X, Y, Z\} + j\{Y, Z, X\} + j^2\{Z, X, Y\} = 0$$

A very simple result seems very significant here: any finite ternary algebraic structure defined above contains and induces the *ordinary Lie algebra structure* if the associative algebra includes the unit element (denoted by 1 here) commuting with all other elements. Here is the proof:

$$\begin{aligned} \{X, 1, Y\} &= (X1Y) + j(1YX) + j^2(YX1) = \\ &= (XY) + j(YX) + j^2(YX) = XY - YX = [X, Y] \end{aligned}$$

because  $j + j^2 = -1$

Here again, the Jacobi identity can be reconstructed if we iterate the same ternary bracket with the unit element employed twice:

$$\begin{aligned} \{\{X, 1, Y\}, 1, Z\} + \{\{Y, 1, Z\}, 1, X\} + \{\{Z, 1, X\}, 1, Y\} &= \\ &= [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \end{aligned}$$

Nevertheless the straightforward ternary generalization of the Jacobi identity is not satisfied if we replace the unity 1 by arbitrary elements  $S$  and  $T$  of the associative algebra:

$$\{\{X, S, Y\}, T, Z\} + \{\{Y, S, Z\}, T, X\} + \{\{Z, S, X\}, T, Y\} \neq 0$$

The Jacobi identity generalizes in fact in two distinct identities:

$$\begin{aligned} & \{Z, \{X, S, T\}, Y\} + \{X, \{S, Y, T\}, Z\} + \{Y, \{S, T, Z\}, X\} + \{S, \{X, Y, Z\}, T\} = \\ & = \{Z, \{X, T, S\}, Y\} + \{X, \{T, Y, S\}, Z\} + \{Y, \{T, S, Z\}, X\} + \{T, \{X, Y, Z\}, S\} \end{aligned}$$

and

$$\begin{aligned} & \{T, \{X, Y, Z\}, S\} + \{T, \{Y, X, S\}, Z\} + \{T, \{Z, S, X\}, Y\} + \{T, \{S, Z, Y\}, X\} = \\ & = \{S, \{Z, X, Y\}, T\} + \{Z, \{S, Y, X\}, T\} + \{Y, \{X, Z, S\}, T\} + \{X, \{Y, S, Z\}, T\} \end{aligned}$$

We can explore the structure of the simplest ternary algebras defined by the abstract ternary bracket displaying the well-defined symmetry property  $\{X, Y, Z\} = j\{Y, Z, X\}$ . We shall start with an example showing how such a non-associative ternary composition rule can be introduced even without any associative algebra behind.

Let us consider the linear space of the *tri-linear forms* over a given N-dimensional vector space E over complex numbers. In a given basis, any such 3-form can be represented by its components  $U_{ABC}$ ,  $A, B.. = 1, \dots, N$ . Suppose also that a metric is defined on E; we can always diagonalize it and choose the basis in which its components form the Kronecker tensor  $\delta_{AB}$ ; its inverse is then  $\delta^{BC}$ . Now, with the help of the metric we can raise the indices, and the following ternary composition rule for the 3-forms may be defined:

$$(U, V, W)_{ABC} = \delta^{EF} U_{FAG} \delta^{GH} V_{HBJ} \delta^{JK} W_{KEC} ;$$

In what follows, we shall just sum over the repeated indices without writing explicitly the metric:

$$(U, V, W)_{ABC} = U_{EAG} V_{GBH} W_{HCE}$$

This is the closest analogue of matrix multiplication that we can imagine, although it is non-associative, what can be readily verified by definition. Now,

in order to come closer to ternary analogue of a Lie algebra, we should form a “ternary commutator” as introduced above:

$$\{U, V, W\}_{ABC} = (U, V, W)_{ABC} + j(V, W, U)_{ABC} + j^2(W, U, V)_{ABC}$$

Obviously, the resulting 3-form displays the following internal symmetry:

$$\{U, V, W\}_{ABC} = j\{U, V, W\}_{BCA} = j^2\{U, V, W\}_{CAB}$$

If we want our ternary  $\mathbb{Z}_3$ -graded algebra to be closed under this new composition law, we should restraint it to the ternary forms having the above symmetry property. Such 3-forms will span a vector space over *real numbers*; its (real) dimension is  $(N^3 - N)/3$ .

The most primitive ternary algebra of such 3-forms is obtained when the indices  $A, B, \dots$  take on two values only, 1 and 2. The only two independent elements of this algebra are the following:

$$\rho_{121}^{(1)} = 1, \quad \rho_{211}^{(1)} = j^2, \quad \rho_{112}^{(1)} = j$$

all other components vanishing, and

$$\rho_{212}^{(2)} = 1, \quad \rho_{122}^{(2)} = j^2, \quad \rho_{221}^{(2)} = j$$

with all other components vanishing.

Then the following ternary algebra is generated by these two 3-forms:

$$\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\} = -\rho^{(2)} \quad \text{and} \quad \{\rho^{(2)}, \rho^{(1)}, \rho^{(2)}\} = -\rho^{(1)}.$$

By definition, the ternary commutator of the same 3-form taken with itself three times is automatically vanishing.

Suppose now that we wish to represent this algebra by means of associative complex matrices, with the ternary composition law defined as

$$\{A, B, C\} = ABC + jBCA + j^2CAB + CBA + j^2BAC + jACB$$

In the case of the ternary algebra of the 3-forms  $\rho^\alpha$  defined above, with  $\alpha = 1, 2$ , complex  $2 \times 2$ -matrices are enough to define the only two elements. It is a simple exercise to prove that such a realization is given by any two Pauli matrices (e.g.  $\sigma_3, \sigma_2$ ) divided by  $\sqrt{2}$ .

It is also possible to represent the eight dimensional ternary algebra obtained when the indices  $A, B, \dots$  take three values by means of  $3 \times 3$  complex matrices. These may be viewed as the generators of the Lie algebra of the unitary group  $SU(3)$ , although in a somewhat unusual basis.

## 9 Ternary generalization of Clifford algebras.

Let us rewrite the usual definition of Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} - 1$$

with  $\mu, \nu = 1, 2, \dots, N$ , in a slightly different way, which will immediately suggest the  $\mathbb{Z}_3$ -graded generalization:

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2g^{\mu\nu} - 1$$

Let us consider an algebra spanned by  $N$  generators  $Q^k$  whose ternary products should satisfy the identity

$$Q^{k_1} Q^{k_2} Q^{k_3} = \varrho_{k_1 k_2 k_3}(\sigma) Q^{k_{\sigma(1)}} Q^{k_{\sigma(2)}} Q^{k_{\sigma(3)}} + 3 \eta^{k_1 k_2 k_3} 1,$$

where  $k_i = 1, \dots, N$ ,  $\sigma$  is the substitution  $(1\ 2\ 3) \in \mathbb{Z}_3$  and  $\varrho_{k_1 k_2 k_3}$  is the representation of the cyclic group  $\mathbb{Z}_3$  by complex numbers which depend on the indices  $k_1, k_2, k_3$  ( $\varrho_{k_1 k_2 k_3}(\sigma) = j$  or  $\varrho_{k_1 k_2 k_3}(\sigma) = j^2$ ) and do not change under the cyclic permutation of the indices, i.e.  $\varrho_{k_1 k_2 k_3} = \varrho_{k_2 k_3 k_1} = \varrho_{k_3 k_1 k_2}$ . If all binary products  $Q^k Q^m$  are linearly independent we shall call the algebra generated by  $Q^1, Q^2, \dots, Q^N$  ternary Clifford algebra. In the case where the above identity for

ternary products can be derived from some binary identities, we shall call the corresponding algebra  $\mathbb{Z}_3$ -graded Clifford algebra.

Applying the above relation three times in a row one obtains the condition on the tensor  $\eta^{klm}$ :

$$\eta^{klm} + \varrho_{klm}(\sigma)\eta^{lmk} + \varrho_{klm}^2(\sigma)\eta^{mkl} = 0.$$

Since  $1 + \varrho_{klm} + \varrho_{klm}^2 = 0$  the above equation has two independent solutions, namely

$$(a): \eta^{klm} = \eta^{lmk} = \eta^{mkl} \text{ and (b): } \eta^{klm} = \varrho_{klm}^2(\sigma)\eta^{lmk} = \varrho_{klm}(\sigma)\eta^{mkl}.$$

The first choice (a) implies the following relation for the generators  $Q^m$ :

$$Q^k Q^l Q^m + Q^l Q^m Q^k + Q^m Q^k Q^l = 3(1 - \varrho_{klm}^2(\sigma)) \eta^{klm} 1,$$

whereas the choice of the solution (b) leads to the relation

$$Q^k Q^l Q^m + Q^l Q^m Q^k + Q^m Q^k Q^l = 0;$$

If  $(k, l, m, n)$  is a set of indices such that  $Q^k Q^l Q^m Q^n \neq 0$ , then the following condition of consistency of representations must be satisfied

$$\varrho_{klm} \varrho_{mkn} \varrho_{lkn} \varrho_{nlm} = 1.$$

In this section we shall construct and explore some examples of the above algebra with the tensor

$$\eta^{klm} = \pm \frac{1}{3}(1 - \varrho_{klm}^2(\sigma))^{-1} \sum_{p=1}^N \delta_p^k \delta_p^l \delta_p^m.$$

The above tensor satisfies the equation (a) and it leads to the algebras whose generators are subjected to the following relations:  $(Q^1)^3 = (Q^2)^3 = \dots = (Q^N)^3 =$

$\pm 1$  and  $Q^k Q^l Q^m = \varrho_{klm}(\sigma) Q^l Q^m Q^k$  if there are at least two different indices in the triple  $(k, l, m)$ .

In the simplest case  $N = 1$  the above described construction leads us to the algebras generated by  $Q_\pm$  which satisfies the relation  $Q_\pm^3 = \pm 1$ . We shall denote the corresponding algebras by  $\mathcal{C}_\pm^1$ . As a vector space each of the algebras  $\mathcal{C}_\pm^1$  is spanned by the monomials  $1, Q_\pm, Q_\pm^2$ . The operators  $Q_+ = \partial_{\theta^2} + \theta$  and  $Q_- = j\partial_\theta + \theta^2$  (where  $\partial_\theta$  is the Grassmann derivation introduced in 3 and  $\partial_{\theta^2}$  gives 1 when applied to  $\theta^2$  and 0 when applied to 1 or  $\theta$ ) acting on the one-dimensional  $\mathbb{Z}_3$ -graded Grassmann algebra with the generator  $\theta$  give the matrix representation of  $\mathcal{C}_+^1$  and  $\mathcal{C}_-^1$ .

In the case  $N = 2$  there are two representations  $\varrho_{211}, \varrho_{122}$  and there is only one consistency condition to be satisfied

$$(\varrho_{211} \varrho_{122})^2 = 1.$$

The above condition shows that the representations  $\varrho_{211}, \varrho_{122}$  should be conjugate to each other, i.e.  $\varrho_{211} = \bar{\varrho}_{122}$ . Let us choose  $\varrho_{122}(\sigma) = j, \varrho_{211}(\sigma) = j^2$ . This choice of representations leads to the following ternary relations

$$Q^2 Q^1 Q^1 = j^2 Q^1 Q^1 Q^2 = j Q^1 Q^2 Q^1 \quad \text{and} \quad Q^1 Q^2 Q^2 = j Q^2 Q^2 Q^1 = j^2 Q^2 Q^1 Q^2.$$

Let us rewrite the above relations in the following form:

$$Q^k (Q^2 Q^1 - j Q^1 Q^2) = 0 \quad \text{and} \quad (Q^2 Q^1 - j Q^1 Q^2) Q^k = 0 \quad \text{with } k = 1, 2.$$

This form suggests that the ternary relations can be derived from the binary ones  $Q^2 Q^1 = j Q^1 Q^2$ . If one associates grade 1 to the generators  $Q^1, Q^2$  then the above binary relation implies that  $Q^1, Q^2$  are  $\mathbb{Z}_3$ -commutative generators, i.e.  $[Q^2, Q^1]_{\mathbb{Z}_3} = 0$ . Thus the algebra generated by the  $\mathbb{Z}_3$ -commutative generators  $Q^1, Q^2$  of grade 1 such that  $(Q^1)^3 = (Q^2)^3 = 1$  is a  $\mathbb{Z}_3$ -graded Clifford algebra

and it provides a realization of the general structure described at the beginning of this section for  $N = 2$ . Let us denote this algebra by  $\mathcal{C}_{\mathbb{Z}_3}^2$ .

Let us assume that  $\mathcal{C}^1(Q^1)$  and  $\mathcal{C}^1(Q^2)$  are two copies of the one-dimensional ternary Clifford algebra generated respectively by the elements  $Q^1$  and  $Q^2$ . We define the  $\mathbb{Z}_3$ -graded tensor product of the algebras  $\mathcal{C}^1(Q^1)$  and  $\mathcal{C}^1(Q^2)$  as the tensor product of the underlying  $\mathbb{Z}_3$ -graded vector spaces endowed with the multiplication defined as follows:

$$(A \otimes B)(A' \otimes B') = j^{gr(B)gr(A')} AA' \otimes BB',$$

where  $A, A' \in \mathcal{C}^1(Q^1)$  and  $B, B' \in \mathcal{C}^1(Q^2)$ . Then  $\mathcal{C}_{\mathbb{Z}_3}^2 = \mathcal{C}^1(Q^2) \otimes_{\mathbb{Z}_3} \mathcal{C}^1(Q^1)$ .

Now we turn to the matrix representation of the algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$ . The matrix algebras generated by a pair of  $n \times n$  matrices  $A$  and  $B$  satisfying relations like  $A^n = B^n = \pm 1$  and  $AB = \mu BA$ , for  $\mu$  a primitive  $n^{\text{th}}$  root of unity were studied by Sylvester [14] in relation with the quaternion-like algebras. If  $n = 3$  the pair of matrices  $A, B$  gives the matrix realization of the  $\mathbb{Z}_3$ -graded Clifford algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$ .

Sylvester called the elements of this nine-dimensional matrix algebra *nonions*. The matrices representing the algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$  are the following ones

$$Q^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Q^2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}$$

where the corresponding operators acting on the Grassmann algebra  $\mathcal{T}^1$  have the form  $Q^1 = \partial_{\theta^2} + \theta$ ,  $Q^2 = \partial_{\theta^2} + j\theta + (j^2 - j)\theta^2\partial_\theta$ . This is made clear by representing an element of the Grassmann algebra with the vector of its components along 1,  $\theta$  and  $\theta^2$ . So, the concept of a ternary Clifford algebra throws light upon nonions from a new point of view and leads to interesting conclusions.

## 10 Ternary analogue of Orthogonal Group

It is well known that Clifford algebras and their matrix representations provide an appropriate framework for the spinor groups. Therefore the natural question is whether there is an analogue of spinor groups covering the group of orthogonal matrices in the case of the  $\mathbb{Z}_3$ -graded Clifford algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$ . It turns out that the answer to this question is positive and the corresponding construction leads to the interesting ternary analogue of orthogonal matrices.

We shall use the ternary forms  $\eta$  and  $\bar{\eta}$  with coefficients transforming according to the laws

$$\eta_{ijk} = j \eta_{jki} = j^2 \eta_{kij} \quad \text{and} \quad \bar{\eta}_{ijk} = j^2 \bar{\eta}_{jki} = j \bar{\eta}_{kij}$$

as analogues of the skew-symmetric matrices. We shall call forms like  $\eta$  the  $j$ -skew-symmetric forms and forms like  $\bar{\eta}$  the  $j^2$ -skew-symmetric forms. Let us denote by  $\mathcal{F}$  the vector space of the  $j$ -skew-symmetric forms and by  $\bar{\mathcal{F}}$  the vector space of the  $j^2$ -skew-symmetric forms. It is clear that each pair of indices  $(i, j)$ ,  $i \neq j$  determines two linearly independent components  $\eta_{ijj}$ ,  $\eta_{jii}$  of the form  $\eta$ . Let us note  $\eta_{211} = \omega$ ,  $\eta_{122} = \chi$ . Then  $\eta_{112} = j^2 \omega$ ,  $\eta_{121} = j \omega$  and  $\eta_{212} = j \chi$ ,  $\eta_{221} = j^2 \chi$ . It is useful in what follows to associate to the form  $\eta$  two vectors and two associated complex numbers  $\omega$  and  $\chi$ :

$$\eta_v = (\eta_{121}, \eta_{112}, \eta_{211}) = \omega \cdot (j, j^2, 1) \quad \text{and} \quad \eta'_v = (\eta_{212}, \eta_{221}, \eta_{122}) = \chi \cdot (j, j^2, 1)$$

and to consider the formal vector  $Q_v = (Q^1, Q^2, Q^3)$  (where  $Q^3 = (Q^1)^2(Q^2)^2$ ) whose components are the grade 1 monomials of the  $\mathbb{Z}_3$ -graded Clifford algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$ . Let us define  $\eta(Q_v) = \omega Q^2 Q^1 Q^1$  and  $\bar{\eta}(Q_v) = \chi Q^1 Q^2 Q^2$ .

Let us now define the map  $m : \mathbf{C}^3 \rightarrow (\mathcal{C}_{\mathbb{Z}_3}^2)_{\bar{1}}$  by the formula

$$m(z) = z_1 Q^1 + z_2 Q^2 + j z_3 Q^3 \quad \text{with } z = (z_1, z_2, z_3) \in \mathbf{C}^3$$

and the linear map  $R$  from the vector spaces  $\mathcal{F}$  and  $\bar{\mathcal{F}}$  to the subalgebra  $(\mathcal{C}_{\mathbb{Z}_3}^2)_{\bar{0}}$  by the formulae

$$R(\eta) = \eta(Q_v) \quad \text{and} \quad R(\bar{\eta}) = \bar{\eta}(Q_v).$$

The following identities are easily verified

$$[R(\eta), m(z)]_{\mathbb{Z}_3} = (j - 1) m(\psi(\eta_v)z), \quad (5)$$

$$[R(\bar{\eta}), m(z)]_{\mathbb{Z}_3} = (j - 1) m(\psi(\eta'_v)z). \quad (6)$$

where the two mappings  $\psi$  and  $\varphi$  are defined by the formulae

$$\psi(x, y, z) = \begin{pmatrix} 0 & 0 & x \\ y & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \quad \text{and} \quad \varphi(x, y, z) = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & z \\ x & 0 & 0 \end{pmatrix}.$$

In analogy with the classical case we consider the subalgebra  $(\mathcal{C}_{\mathbb{Z}_3}^2)_{\bar{0}}$  of the elements of grade 0 of the algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$ . It is spanned by the monomials 1,  $Q^2Q^1Q^1$  and  $Q^1Q^2Q^2$ . Let us introduce the notations  $T = Q^2Q^1Q^1$  and  $S = Q^1Q^2Q^2$ . The structure of the subalgebra  $(\mathcal{C}_{\mathbb{Z}_3}^2)_{\bar{0}}$  is completely described by the formulae

$$T^{3k} = 1, \quad T^{3k+1} = T, \quad T^{3k+2} = S, \quad S^{3k} = 1, \quad S^{3k+1} = S, \quad S^{3k+2} = T$$

Let us consider the elements of the subalgebra  $(\mathcal{C}_{\mathbb{Z}_3}^2)_{\bar{0}}$  which have the form of Gaussian expressions

$$e^{R(\eta)} = e^{\eta(Q_v)}$$

Let us denote by  $\mathcal{S}$  the group generated by the above elements. This group is commutative and any of its elements can be expressed as follows

$$g(\omega) = \Theta(\omega) + \Psi(\omega) T + \Phi(\omega) S$$

where the coefficients are the functions

$$\Theta(\omega) = \frac{1}{3}(e^\omega + e^{j\omega} + e^{j^2\omega}), \quad \Psi(\omega) = \frac{1}{3}(e^\omega + j e^{j^2\omega} + j^2 e^{j\omega}), \quad \Phi(\omega) = \frac{1}{3}(e^\omega + j e^{j\omega} + j^2 e^{j^2\omega})$$

If  $g(\omega)$  and  $g(\omega)'$  are two elements of  $\mathcal{S}$  then

$$g(\omega)g(\omega') = g(\omega + \omega').$$

It turns out that just as in the case of the classical Clifford algebra the identities (5) and (6) can be realized on the group level and we shall construct this realization only for the first one since the realization of the second one can be done similarly.

Let us consider the functional  $3 \times 3$ -matrices depending on complex variable  $\omega$

$$A(\omega) = \begin{pmatrix} \alpha(\omega) & \beta(\omega) & \lambda(\omega) \\ j^2\lambda(\omega) & \alpha(\omega) & j\beta(\omega) \\ j^2\beta(\omega) & j\lambda(\omega) & \alpha(\omega) \end{pmatrix}$$

where the entries are the functions

$$\begin{aligned} \alpha(\omega) &= \frac{1}{3}(e^\omega e^{-j^2\omega} + e^{j\omega} e^{-\omega} + e^{j^2\omega} e^{-j\omega}), \\ \beta(\omega) &= \frac{1}{3}(e^\omega e^{-j^2\omega} + j e^{j\omega} e^{-\omega} + j^2 e^{j^2\omega} e^{-j\omega}), \\ \lambda(\omega) &= \frac{1}{3}(j e^\omega e^{-j^2\omega} + e^{j\omega} e^{-\omega} + j^2 e^{j^2\omega} e^{-j\omega}). \end{aligned}$$

Straightforward computations show that

$$A(\omega)A(\omega') = A(\omega + \omega') \quad \text{and} \quad \det A(\omega) = 1.$$

Consequently the above one-parameter matrices form the commutative group we shall denote by  $\mathcal{S}^*$ . We let  $\pi : \mathcal{S} \rightarrow \mathcal{S}^*$  be the map

$$\pi(g(\omega)) = A(\omega),$$

where  $g(\omega)$  is an element of  $\mathcal{S}$ . One can see that  $\pi$  is a homomorphism of groups.

It can be proved that

$$g^{-1}(\omega) m(z) g(\omega) = m(\pi(g(\omega))z),$$

and the above formula gives the realization of the formula 5 on the group level.

Let  $A$  be an arbitrary  $3 \times 3$ -matrix

$$A = \begin{pmatrix} \alpha & \beta^* & \tilde{\lambda} \\ \tilde{\alpha}_1 & \beta_1 & \lambda_1^* \\ \alpha_2^* & \tilde{\beta}_2 & \lambda_2 \end{pmatrix}.$$

We define the *cyclic transposition*  $t_c$  of the  $3 \times 3$ -matrix  $A$  as follows: the entries marked with asterisk undergo the counterclockwise cyclic permutation and the entries marked with tilde undergo the clockwise cyclic permutation. Applying this definition to the matrix  $A$  one gets

$$A^{t_c} = \begin{pmatrix} \alpha & \lambda_1^* & \tilde{\alpha}_1 \\ \tilde{\beta}_2 & \beta_1 & \alpha_2^* \\ \beta^* & \tilde{\lambda} & \lambda_2 \end{pmatrix}.$$

Note that the cyclic transposition does not change the trace of a matrix and  $A^{t_c^3} = A$ , where  $t_c^3$  means the cyclic transposition applied three times. Then it can be shown that the one-parameter matrices  $A(\omega)$  satisfy the condition

$$A(\omega) A^{t_c}(\omega) A^{t_c^2}(\omega) = \mathbf{Id}.$$

The above stated condition can be considered as the ternary generalization of the classical orthogonality and the group  $\mathcal{S}^*$  can be considered as the analogue of the group of orthogonal matrices. Taking the matrix  $A$  in the exponential form  $A = e^B = \mathbf{Id} + B + \frac{1}{2!}B^2 + \dots$  one can get the infinitesimal form of the ternary orthogonality, that is

$$B + B^{t_c} + B^{t_c^2} = 0.$$

Note that the matrices  $\psi(\eta_v)$  associated with a  $j$ -skew-symmetric form  $\eta$  satisfy above condition.

The matrix representation of the algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$  allows to establish the analogue of the special case of the Mathai-Quillen formula [15]. Let  $A$  be a skew-symmetric  $2 \times 2$ -matrix

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

Then the Mathai-Quillen formula in the case of the algebra  $\mathcal{C}^2$  can be written as follows:

$$\text{Str}(e^{\frac{1}{2}\gamma^t A \gamma}) = 2i\left(\frac{\sinh ia}{ia}\right) \sqrt{\det A}.$$

The grade 0 monomials  $T, S$  of the algebra  $\mathcal{C}_{\mathbb{Z}_3}^2$  have the following matrix representation :

$$T = Q^2 Q^1 Q^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad S = Q^1 Q^2 Q^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}.$$

Note that these matrices determine two different  $\mathbb{Z}_3$ -structures on the complex space  $\mathbf{C}^3$ . Let us choose the  $\mathbb{Z}_3$ -structure defined by the matrix  $T$ . We replace the supertrace used in the Mathai-Quillen formula by the notion of the  $\mathbb{Z}_3$ -graded trace (hypertrace) defined in section 5. Now the analogue of the Mathai-Quillen formula has the form

$$\text{tr}_{\mathbb{Z}_3}(e^{\eta(P_v)}) = 3\omega^{-1} \Phi(\omega) \sqrt[3]{\det(\psi(\eta_v))}$$

For a more detailed exposition, see [16].

## 11 $\mathbb{Z}_3$ -analogue of the Dirac equation

The  $\mathbb{Z}_3$ -graded generalization of Grassmann algebra discussed in section 3 leads to a natural generalization of the superfields as the fields composed of different contributions proportional to all possible monomials in the  $\mathbb{Z}_3$ -graded generators  $\theta^A$  and  $\bar{\theta}^B$ , i.e.

$$\Phi(\theta^A, \bar{\theta}^B, x^\mu) = \phi_0(x^\mu) + \psi_A(x^\mu)\theta^A + \bar{\theta}^B\bar{\psi}_B + \chi_{AB}\theta^A\bar{\theta}^B + \dots$$

However, this approach is made in the context of the *second quantization*, with operator-valued fields. One could ask if the ternary character of such a theory could not be perceived even at a deeper level, i.e. in the algebraic properties of the complex valued wave functions which would be the solutions of some Schrödinger-like differential equations of a new type.

The usual (binary) Clifford algebra appears in a natural manner in Dirac's equation which is, in a sense, a "square root" of the Klein-Gordon equation. With the use of the ternary Clifford algebra defined above, the  $\mathbb{Z}_3$ -graded generalization of Dirac's equation should read:

$$\frac{\partial \psi}{\partial t} = Q^1 \frac{\partial \psi}{\partial x} + Q^2 \frac{\partial \psi}{\partial y} + Q^3 \frac{\partial \psi}{\partial z} + T m \psi$$

where  $\psi$  stays for a triplet of wave functions, which can be considered either as a column, or as a grade 1 matrix with three non-vanishing entries  $u \ v \ w$ ,

$$Q^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q^2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, Q^3 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}$$

and  $T$  is the diagonal  $3 \times 3$  matrix with the eigenvalues 1,  $j$  and  $j^2$ . It is interesting to note that this is possible only with *three* spatial coordinates.

In order to diagonalize this equation, we must act three times with the same operator, which will lead to the equation of *third* order, satisfied by each of the three components  $u, v, w$ , e.g.:

$$\frac{\partial^3 u}{\partial t^3} = \left[ \frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - 3 \frac{\partial^3}{\partial x \partial y \partial z} \right] u + m^3 u$$

This equation can be solved by separation of variables; the time-dependent and the space-dependent factors are linear combinations of  $\Theta(\omega t)$ ,  $\Psi(\omega t)$ ,  $\Phi(\omega t)$  and  $\Theta(\mathbf{k.r})$ ,  $\Psi(\mathbf{k.r})$ ,  $\Phi(\mathbf{k.r})$ . Their nine independent products can be represented in a basis of real functions as

$$\begin{pmatrix} A_{11} e^{\omega t + \mathbf{k.r}} & A_{12} e^{\omega t - \frac{\mathbf{k.r}}{2}} \cos \xi & A_{13} e^{\omega t - \frac{\mathbf{k.r}}{2}} \sin \xi \\ A_{21} e^{-\frac{\omega t}{2} + \mathbf{k.r}} \cos \tau & A_{22} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \cos \tau \cos \xi & A_{23} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \cos \tau \sin \xi \\ A_{31} e^{-\frac{\omega t}{2} + \mathbf{k.r}} \sin \tau & A_{32} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \sin \tau \cos \xi & A_{33} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \sin \tau \sin \xi \end{pmatrix}$$

where  $\tau = \frac{\sqrt{3}}{2}\omega t$  and  $\xi = \frac{\sqrt{3}}{2}\mathbf{k} \cdot \mathbf{r}$ .

The parameters  $\omega$ ,  $\mathbf{k}$  and  $m$  must satisfy the cubic dispersion relation:  $\omega^3 = k_x^3 + k_y^3 + k_z^3 - 3k_x k_y k_z + m^3$ . Although neither of the solutions belong to the space of tempered distributions, it is possible to combine them into solutions of the ordinary Klein-Gordon equation: The ternary skew-symmetric products contain only trigonometric functions, depending on the combinations  $2(\tau - \xi)$  and  $2(\tau + \xi)$ . As a matter of fact, not only the *determinant*, but also each of the *minors* of this matrix is a combination of the trigonometric functions only. The same is true for the binary products of “conjugate” solutions, with opposite signs for  $\omega t$  and  $\mathbf{k} \cdot \mathbf{r}$  in the exponentials. It is possible to find new parameters, which are linear combinations of  $\omega$ ,  $\mathbf{k}$  and  $m$ , that will satisfy quadratic relations that may be interpreted as a mass shell equation.

## 12 Conclusion

It is clear from these examples that most of the mathematical structures that are commonly used in supersymmetric theories can be generalized from the  $\mathbb{Z}_2$ -graded case to the  $\mathbb{Z}_3$ -graded case. These generalizations, though, are very different in their spirit from other generalizations known as fractional supersymmetry.

The most important difference is that *ternary* rather than *binary* relations define the algebraic structures of functions and fields.

Moreover, the ternary principle is extended in a natural way to all the algebraic structures used in the classical field theories, such as Grassmann, Lie and Clifford algebras.

The obvious inspiration for investigating this type of mathematical structures comes from the idea that the confinement of quarks should have its origin in the special algebraic structure of the corresponding fields, which might elude the usual rules of quantum field theory, as well as the observation in the form of a

free field. We hope that further work in this direction will contribute to shed some light on these problems.

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